Fictitious domain methods for fracture models in elasticity.

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AGU fall meeting 2014 – Number: DI11A-4240

[1] A linear elasticity model

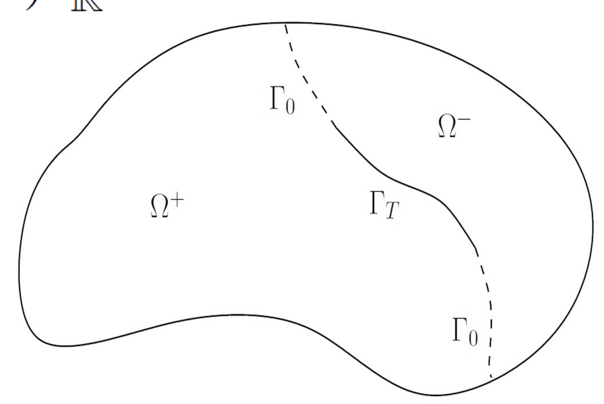
$$\begin{cases} -\mathrm{div}\ \sigma_L(\mathbf{u}) = \mathbf{f} & \text{in } \Omega^+ \cup \Omega^-, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ (\sigma_L(\mathbf{u})\mathbf{n})^\pm = p\mathbf{n}^\pm & \text{on } \Gamma_T, \\ [\mathbf{u}] = 0 & \text{across } \Gamma_0 = \Gamma \setminus \Gamma_T. \end{cases} \qquad \Omega = \Omega^+ \cup \Omega^-$$

Expression of the Lamé stress tensor:

$$\sigma_L(\mathbf{u}) = 2\mu_L \varepsilon(\mathbf{u}) + \lambda_L(\mathrm{div}\ \mathbf{u})\mathrm{I}_{\mathbb{R}^d}$$

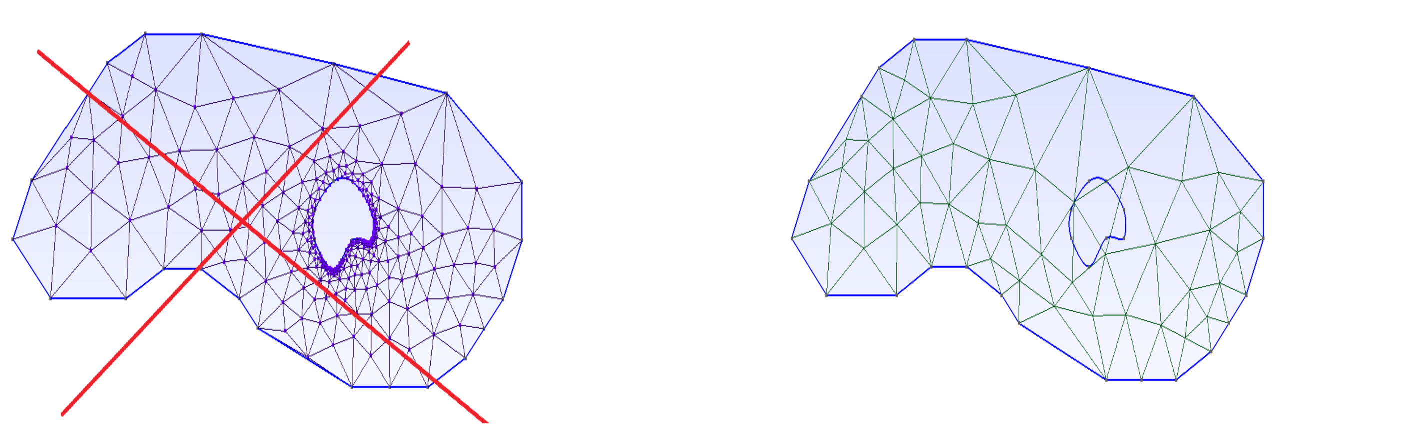
In a weak formulation:

- Right-hand side: the traction forces on the fracture, and some additional forces f.
- The jump/transmission condition: Imposed by a Lagrange multiplier.
- Possibility of taking into account heterogeneity and anisotropy.

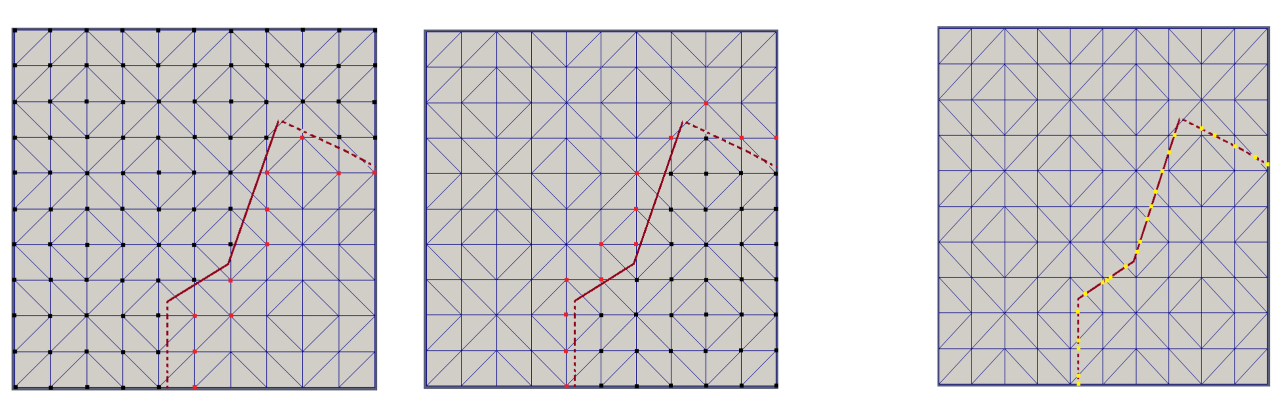


[2] The fictitious domain approach

Use of non-conforming meshes



Selection of degrees of freedom



For the displacements

For the variables on the fracture and its extension

Finite element spaces

Abstract spaces:

$$\begin{aligned} \mathbf{V}^+ &= \{\mathbf{v} \in \mathbf{H}^1(\Omega^+) \mid \mathbf{v} = 0 \text{ on } \partial\Omega \cap \partial\Omega^+\}, \\ \mathbf{V}^- &= \{\mathbf{v} \in \mathbf{H}^1(\Omega^-) \mid \mathbf{v} = 0 \text{ on } \partial\Omega \cap \partial\Omega^-\}, \\ \mathbf{W} &= \mathbf{H}^{-1/2}(\Gamma_0) = \left(\mathbf{H}^{1/2}(\Gamma_0)\right)'. \end{aligned}$$

Discrete spaces: $\tilde{\mathbf{V}}_h \subset \mathbf{H}^1(\Omega), \qquad \tilde{\mathbf{W}}_h \subset \mathbf{L}^2(\Omega),$

$$\mathbf{V}_h^+ := \tilde{\mathbf{V}}_{h|\Omega^+}, \quad \mathbf{V}_h^- := \tilde{\mathbf{V}}_{h|\Omega^-}, \quad \mathbf{W}_h := \tilde{\mathbf{W}}_{h|\Gamma_0}.$$

Form of the global linear system to be solved

$$\left(\begin{array}{cc|c} A^+ & 0 & B^{+T} \\ 0 & A^- & -B^{-T} \\ \hline B^+ & -B^+ & 0 \end{array}\right) \left(\begin{array}{c} \mathbf{U}^+ \\ \mathbf{U}^- \\ \hline \mathbf{\Lambda} \end{array}\right) = \left(\begin{array}{c} \mathbf{F}^+ \\ \mathbf{F}^- \\ \hline 0 \end{array}\right)$$

References:

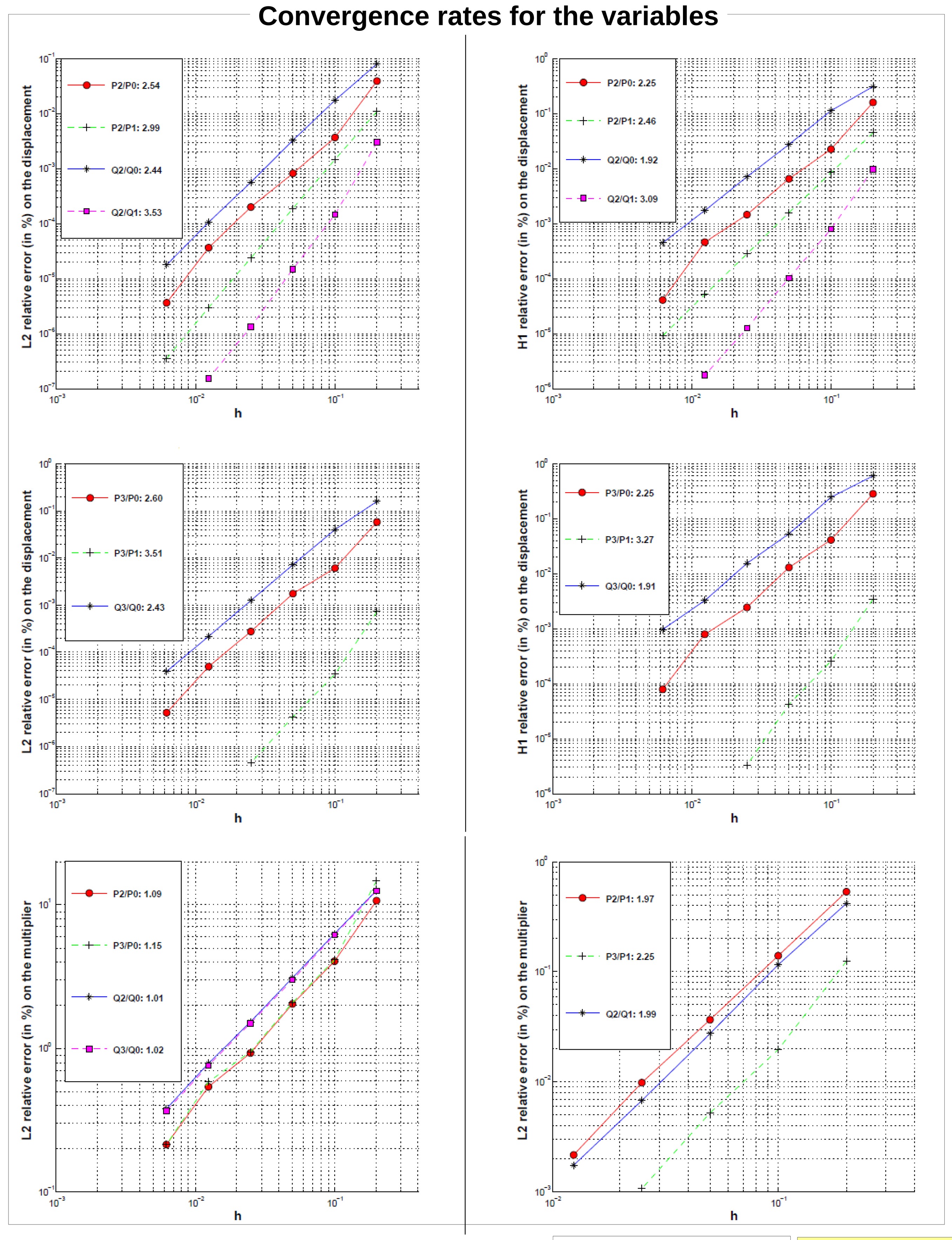
- Fukushima & Al., JGR Solid Earth 2005: *Finding realistic dike models from interferometric synthetic aperture radar data: The February 2000 eruption at Piton de la Fournaise.*
- Haslinger & Renard, SIAM JNA 2009: *A new fictitious domain approach inspired by the eXtended Finite Element Method.*
- Pollard & Al., Tectonophysics 1983: *Surface deformation in volcanic Rift Zones.*

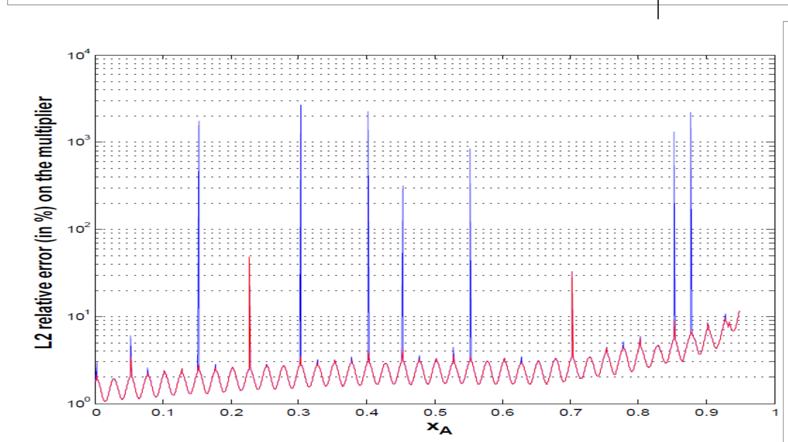
Abstract

We present a generic Finite Element Method designed for pressurized or sheared cracks inside a linear elastic medium. A fictitious domain method is used to take the crack into account independently of the mesh. Besides the possibility of considering heterogeneous media, the approach permits the evolution of the crack through time or more generally through iterations: The goal is to change the less things we need when the crack geometry is modified; In particular no re-meshing is required (the boundary conditions at the level of the crack are imposed by Lagrange multipliers), leading to a gain of computation time and resources with respect to classic finite element methods. This method is also robust with respect to the geometry, since we expect to observe the same behavior whatever the shape and the position of the crack are. We present numerical experiments which highlight the accuracy of our method (using convergence curves), the optimality of errors, and the robustness with respect to the geometry (with computation of errors on some quantities for all kind of geometric configurations). We also provide 2D benchmark tests. The method is then applied to *Piton de la Fournaise* volcano, considering a pressurized crack - inside a 3-dimensional domain - and the corresponding computation time and accuracy is expected to be compared with results from a mixed Boundary element method. In order to determine the crack geometrical characteristics, and pressure, inversions will be performed combining fictitious domain computations with a near neighborhood algorithm. The aim is to compare performances with those obtained combining a mixed boundary element method with the same inversion algorithm.

[3] Numerical tests

Convergence rates for the variables





Robustness with respect to the geometry

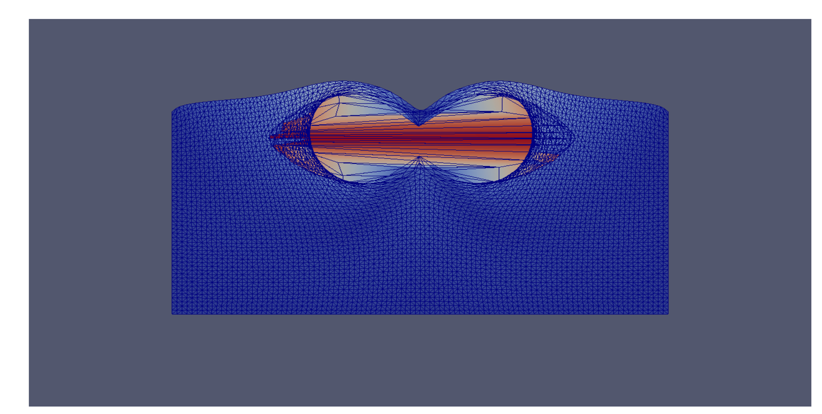
Blue: without stabilization
Red: with Stabilization

Cf: Barbosa & Hughes, 1991-1992.

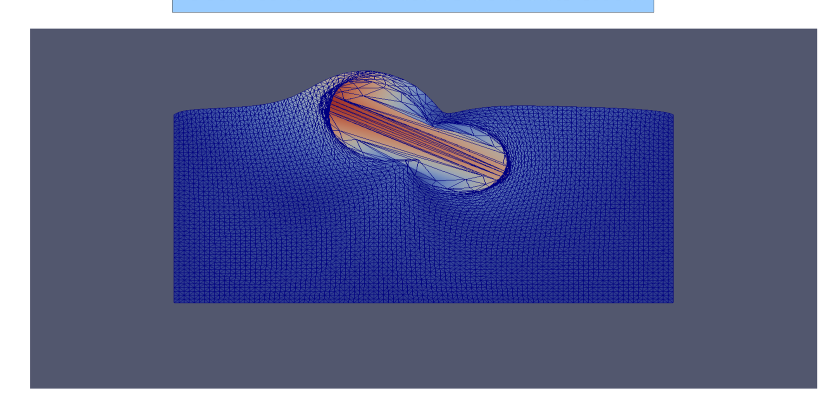
h: mesh size

x_A: position of an inclined straight line

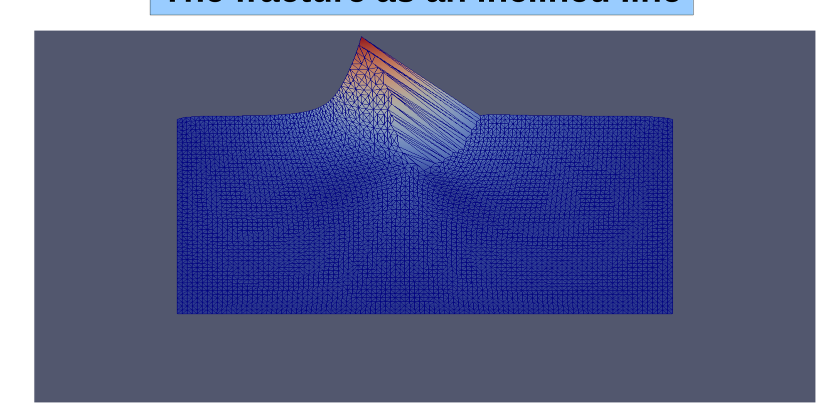
Standard 2D tests



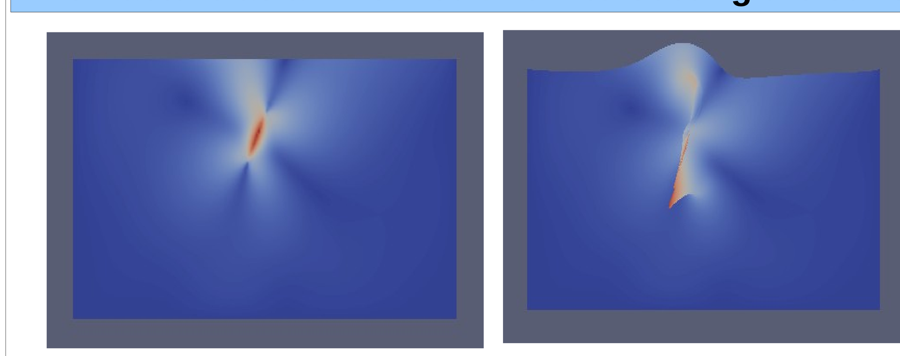
The fracture as vertical line



The fracture as an inclined line



The fracture as an inclined line intersecting the surface



Tangential forces applied on inclined fracture

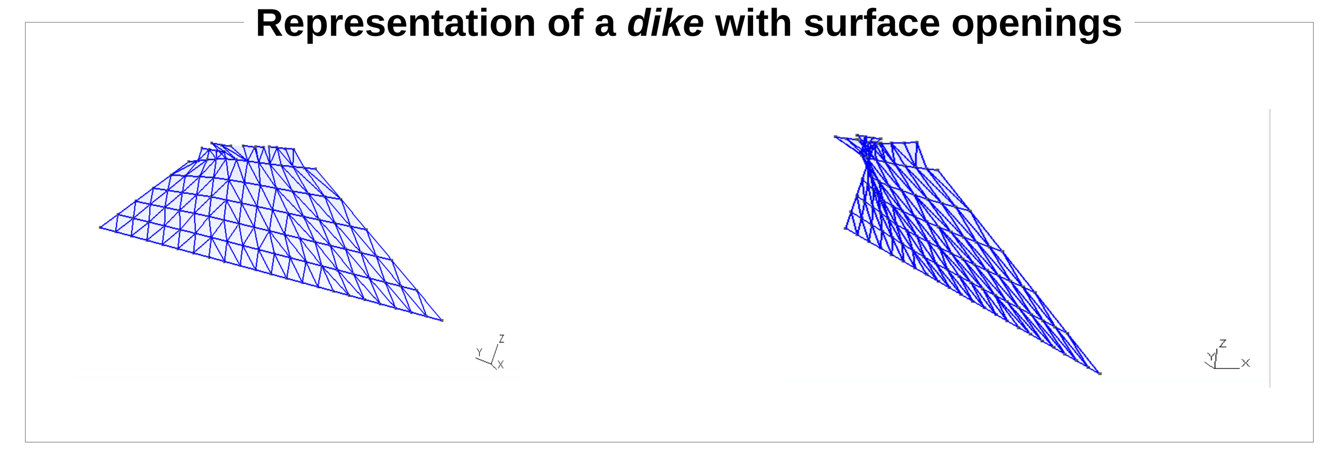
Cf: Pollard & Al., 1983.

Acknowledgments: This research was financed by the French Government Laboratory of Excellence initiative n°ANR-10-LABX-0006, the Région Auvergne and the European Regional Development Fund. This is Laboratory of Excellence ClerVolc contribution number 131.

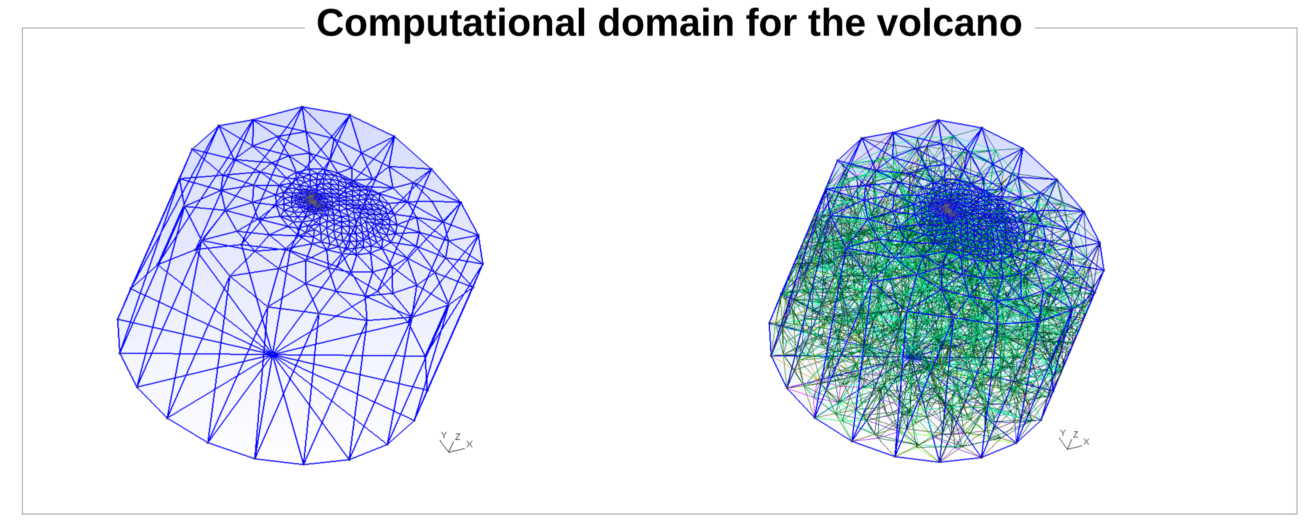
[4] Realistic 3D simulations

Example of *Piton de la Fournaise* (île de la Réunion) February 2000 eruption

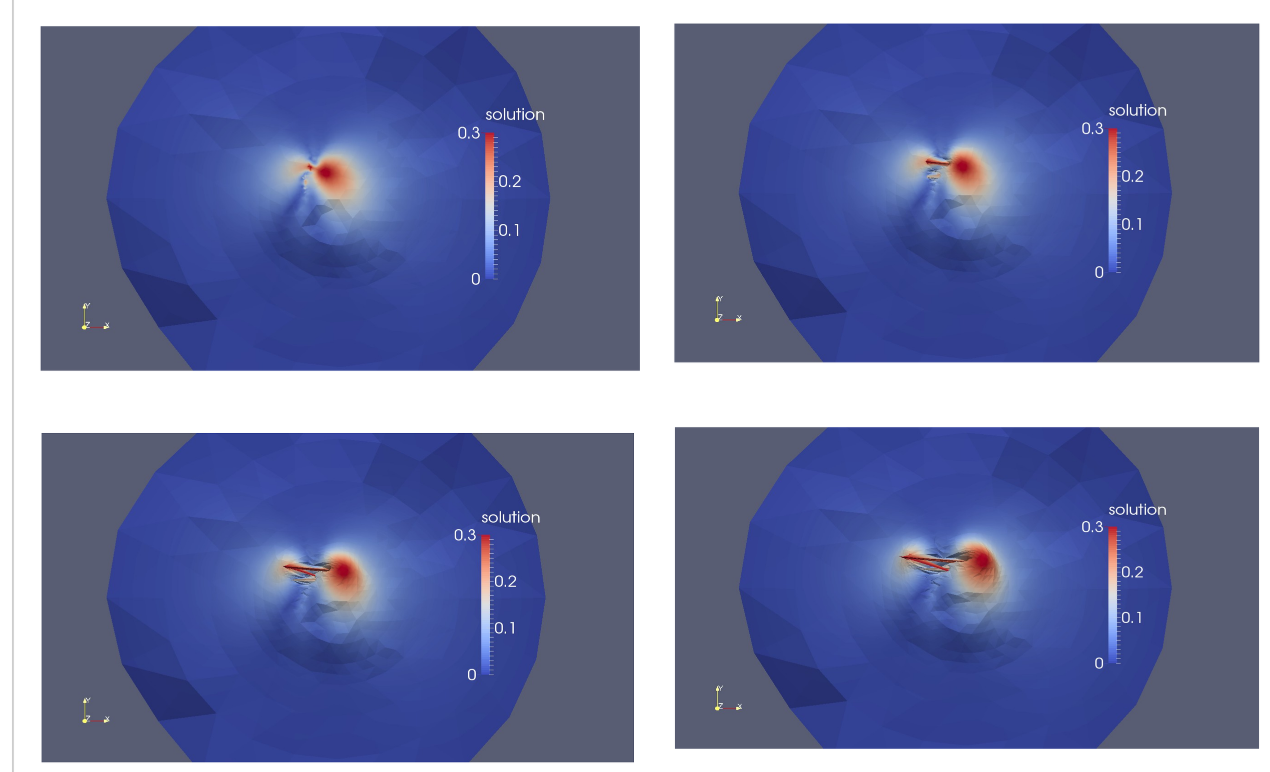
Representation of a *dike* with surface openings



Computational domain for the volcano



Results: Warping with respect to the computed deformation



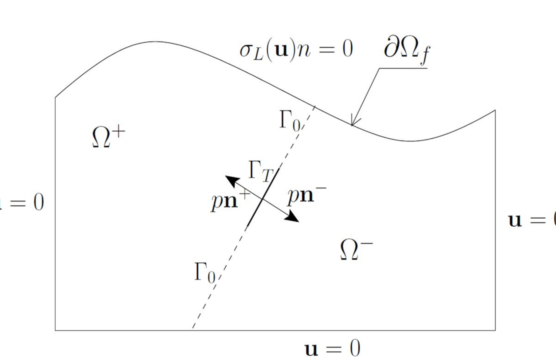
Next step: Considering non-constant elasticity coefficients Obtained experimentally by seismic tomography.

[5] Inverse problem

Finding the crack from surface measurements

Misfit functional:

$$J(\Omega) = \frac{1}{2}\int_{\partial\Omega_f} |\mathbf{u}_{|\partial\Omega_f} - \mathbf{u}_{obs}|^2$$



Monte-Carlo methods:

Principles:
- Computing the misfit cost – with respect to a displacement observation at the surface - for a set of fractures determined by 7 parameters.
- Use of near neighborhood inversion with the fracture parameterization to minimize the misfit function.

Shape optimization methods:

- Writing a gradient of the misfit function with respect to the fracture seen as an abstract object of infinite dimension.
- Discrete parameterization of the fracture: Largest range of degrees of freedom for the geometry of the fracture.
- Performing a gradient algorithm for minimizing the misfit functional.